## 5.4 Steady irrotational flow in 2D, Complex Velocity

In this section we write $(x_1, x_2) = (x, y) \in \mathbb{R}^2$ and consider planar velocity fields of the form

$$\mathbf{v}(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \\ 0 \end{pmatrix}, \tag{5.9}$$

where $u, v$ are sufficiently smooth functions.

Given a 2D velocity field, we form the complex velocity function

$$w(z) = u(x,y) - iv(x,y) \text{ for } z = x + iy \in \mathbb{C}. \tag{5.10}$$

Now consider the complex contour integral around the closed simple contour $\gamma$ parametrised by $s \in [a,b]$.

$$\int_\gamma w(z)\, dz = \int_a^b (u - iv)\left(\frac{dx}{ds} + i\frac{dy}{ds}\right) ds = \int_a^b u\frac{dx}{ds} + v\frac{dy}{ds}\, ds + i\int_a^b u\frac{dy}{ds} - v\frac{dx}{ds}\, ds, \tag{5.11}$$

hence the real part of the integral is the circulation around the contour $\gamma$ and the imaginary part is related to the flow perpendicular to $\gamma$. (Note that $\begin{pmatrix} \frac{dy}{ds} \\ -\frac{dx}{ds} \end{pmatrix}$ is normal[7] to the contour $\gamma$ and so the imaginary part of the integral (5.11) is the 'volume' flow rate[8] across $\gamma$.) Notice, in particular, that if $\gamma$ is a streamline, then the integrand in the imaginary part of (5.11) is zero since the velocity field (evaluated on $\gamma$)is tangent to $\gamma$.

**Remark 5.10.**

*1. The real part of* (5.11) *gives the circulation of the flow around $\gamma$.*

*2. The integrand in imaginary part of the integral* (5.11) *is zero if $\gamma$ is a streamline or the boundary of a body $B$ immersed in the flow (so that $\mathbf{v}.\mathbf{n} = 0$ on $\partial B = \gamma$, which implies that $\partial B$ is a streamline).*

*3. If $s \in [a,b]$ is arc-length, then the imaginary part of* (5.11) *gives the total volume flow rate per unit length across $\gamma$ (viewing $\gamma$ as the cross-section of a prismatic cylinder with axis along the $x_3$ (i.e., $z$) direction).*

*4. If $\mathbf{v}$ corresponds to an incompressible, irrotational flow (so that $\nabla.\mathbf{v} = 0$, $\nabla \times \mathbf{v} = \mathbf{0}$) then $w(z)$ is analytic (and the contour integral on the left hand side of* (5.11) *can be deformed through any region in which $W(z)$ is analytic without changing its value). To see this, suppose that the flow is incompressible, then*

$$0 = \nabla.\mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial}{\partial y}(-v) \tag{5.12}$$

*and if the flow is irrotational, then*

$$\mathbf{0} = \nabla \times \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{pmatrix} \Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial}{\partial x}(-v) \tag{5.13}$$

[7]If $s \in [a,b]$ is an arc length parameterisation, then this is an outward unit normal vector to $\gamma$.

[8]If we view $\gamma$ as the cross-section of a 3D cylindrical domain parallel to the $z-$axis, then this is the volume flow rate per unit length.

*and so the real and imaginary parts of the complex velocity satisfy the Cauchy-Riemann equations of complex analysis. Hence, $w(z)$ is a complex analytic function on its domain of definition.*

*5. On any simply connected domain (and always on any sufficiently small open disc around any point), if $w(z)$ is analytic then there exists an analytic function $\Phi(z)$ such that $\Phi'(z) = w(z)$.*

**Example 5.11** (Potential flows)**.**

*1. Let $\alpha > 0$ and let $\Phi(z) = \alpha \log(z) = \alpha (\ln(|z|) + i \arg(z))$ in the cut plane. Then $w(z) = \Phi'(z) = \frac{\alpha}{z}$. This represents a source at the origin.*

*2. Let $\Phi(z) = \frac{-i\Gamma_0}{2\pi} \log(z)$, then $w(z) = \Phi'(z) = \frac{-i\Gamma_0}{2\pi z}$.*

*This represents a point vortex at the origin with circulation $\Gamma_0$ around the origin.*

*3. Let $\Phi(z) = U_0 \left(z + \frac{a^2}{z}\right)$ so that $w(z) = U_0 \left(1 - \frac{a^2}{z^2}\right)$.*

*This represent flow past a circular cylinder with centre at the origin.*

*See problems sheet 7 for further details.*

**Remark 5.12** (Velocity potential and stream function)**.** *If the flow is irrotational, then $\nabla \times \mathbf{v} = 0$ and so*

$$\mathbf{v}(x, y) = \begin{pmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ 0 \end{pmatrix}, \tag{5.14}$$

*where $\phi(x, y)$ is the (real) velocity potential.*

*If we suppose further that the flow is incompressible, then*

$$0 = \nabla.\mathbf{v} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \Delta \phi, \tag{5.15}$$

*and so $\phi$ is harmonic.*

*Notice that if we do not assume that the flow is irrotational, then by a result from vector calculus, the incompressibility condition $\nabla.\mathbf{v} = 0$ implies the existence of a scalar function $\psi$ called the stream function such that*

$$\nabla \times \begin{pmatrix} 0 \\ 0 \\ \psi \end{pmatrix} = \begin{pmatrix} \psi_y \\ -\psi_x \\ 0 \end{pmatrix}$$

*and hence*

$$\nabla \times \mathbf{v} = \nabla \times \begin{pmatrix} \psi_y \\ -\psi_x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\Delta \psi \end{pmatrix}$$

*is the corresponding vorticity. In particular, if the flow is also irrotational, then $\psi$ is harmonic.*

## 5.5 Flow past an immersed body

Consider flow in a 2D domain external to a connected set $B$ (corresponding to a body immersed in the flow). Consider a 2D velocity field and form the complex velocity function

$w(z)$. If we parametrise the boundary of $B$ as $(x(s), y(s))$ using arc length $s \in [a, b]$ as the parameter, then the vector field

$$\begin{pmatrix} \frac{dx}{ds} \\ \frac{dy}{ds} \end{pmatrix} \tag{5.16}$$

is the unit tangent vector to $\partial D$ and hence

$$\begin{pmatrix} \frac{dy}{ds} \\ -\frac{dx}{ds} \end{pmatrix} \tag{5.17}$$

is the outward pointing normal unit vector to $B$ on $\partial B$.

On the boundary of the body $B$ there is no normal component of velocity and so[9]

$$0 = \mathbf{v}.\mathbf{n} = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} . \begin{pmatrix} \frac{dy}{ds} \\ -\frac{dx}{ds} \end{pmatrix} . \tag{5.18}$$

Now, by (5.11) and Cauchy's Theorem, the complex contour integral

$$\int_\gamma w(z)\,dz = \int_{\partial B} w(z)\,dz = \int_a^b u\frac{dx}{ds}+v\frac{dy}{ds}\,ds+i\int_a^b u\frac{dy}{ds}-v\frac{dx}{ds}\,ds = \text{Circulation around } \partial B. \tag{5.19}$$

**Theorem 5.13** (Blasius Theorem)**.** *Let $B$ denote the region occupied by a body immersed in a steady 2D irrotational, incompressible flow. Let*

$$\mathbf{F} = \begin{pmatrix} F_x \\ F_y \end{pmatrix}$$

*be the force exerted by the fluid on the body. Let $w(z)$ be the complex velocity corresponding to the flow. Then*

$$F_x - iF_y = \frac{i\rho_0}{2} \oint_{\partial B} (w(z))^2\, dz.$$

*Proof.* By (5.11) and since

$$u\frac{dy}{ds} - v\frac{dx}{ds} = 0 \quad \text{on } \partial B \tag{5.20}$$

(note $\mathbf{v}.\mathbf{n} = 0$ on $\partial B$, so that $\partial B$ is a streamline), it follows that on $\partial B$

$$w(z)\,\frac{dz}{ds} = u(x(s), y(s))\frac{dx(s)}{ds} + v(x(s), y(s))\frac{dy(s)}{ds}\,ds.$$

Hence on $\partial B$:

$$\begin{aligned} (w(z))^2 \frac{dz(s)}{ds} &= (u - iv)\left(u\frac{dx}{ds} + v\frac{dy}{ds}\right) \\ &= u^2\frac{dx}{ds} + uv\frac{dy}{ds} - i\left(uv\frac{dx}{ds} + v^2\frac{dy}{ds}\right) \end{aligned}$$

[9] Hence $\partial B$ is a streamline.

and using (5.20) now yields

$$= (u^2 + v^2)\left(\frac{dx}{ds} - i\frac{dy}{ds}\right). \tag{5.21}$$

By an earlier discussion (see Remark 4.7), the force on the body is given by

$$\mathbf{F} = \begin{pmatrix} F_x \\ F_y \end{pmatrix} = \int_{\partial B} -P\mathbf{n}\, ds = \int_a^b -P \begin{pmatrix} \frac{dy}{ds} \\ -\frac{dx}{ds} \end{pmatrix} ds,$$

where $s$ denotes arc-length and we have used the fact that the Cauchy stress tensor is given by $T = -PI$ for an ideal fluid. Hence

$$F_x - iF_y = -\int_a^b P\left(\frac{dy}{ds} + i\frac{dx}{ds}\right) ds = \frac{1}{2} i\rho_0 \int_{\partial B} (u^2 + v^2)\left(\frac{dx}{ds} - i\frac{dy}{ds}\right) ds,$$

where we have used Bernoulli's Theorem that $\frac{P}{\rho_0} + \frac{1}{2}\left(u^2 + v^2\right) = H_0$ (a constant) and that the integral of the constant $H_0$ around a closed contour is zero. Comparing this expression with (5.21) completes the proof. □

**Remark 5.14.** *We can use Blasius Theorem together with the residue theorem to compute the force on objects immersed in a flow.*

**Remark 5.15** (D'Alembert's Paradox.)**.** *This follows on applying Blasius' Theorem to the flow in part 3 of Example 5.11. The contour integral in Blasius' Theorem can be calculated using Cauchy's Residue Theorem and it can be shown that the total force on the cylindrical body due to the flow is zero.*

**Remark 5.16.** *We can add together the complex velocity potentials for simple flows to produce more complicated flows. For example,*

$$\Phi(z) = U_0\left(z + \frac{a^2}{z}\right) - \frac{i\Gamma_0}{2\pi}\log(z),$$

*with $a > 0$, $\Gamma_0 > 0$, represents flow around a cylinder of radius $a$ with circulation $\Gamma_0$ around the disc of radius $a$ (considered as the cross-section of the cylinder). In this case Blasius' Theorem shows that there is a lift force on the cylinder per unit length due to the circulation.*

# 6 Structure of the Euler Equations

Our approach in this section is take the incompressible Euler equations

$$\frac{\partial v_i}{\partial t}(\mathbf{x},t) + v_k(\mathbf{x},t)\frac{\partial}{\partial x_k}v_i(\mathbf{x},t) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x_i}(\mathbf{x},t) \tag{6.1}$$

$$\frac{\partial v_k}{\partial x_k}(\mathbf{x},t) = 0 \tag{6.2}$$

and differentiate the momentum balance equations with respect to $x_j$ to yield

$$\frac{\partial^2 v_i}{\partial t \partial x_j}(\mathbf{x},t) + \frac{\partial v_k}{\partial x_j}\frac{\partial v_i}{\partial x_k} + v_k \frac{\partial}{\partial x_k}\left(\frac{\partial v_i}{\partial x_j}\right) = -\frac{1}{\rho_0}\frac{\partial^2 P}{\partial x_i \partial x_j}(\mathbf{x},t).$$

If we write $\Gamma = (\Gamma_{ij}) = \left(\frac{\partial v_i}{\partial x_j}(\mathbf{x},t)\right)$ for the velocity gradient tensor, then we see that $\Gamma$ satisfies

$$\frac{D\Gamma_{ij}}{Dt} + \left(\Gamma^2\right)_{ij} = -\frac{1}{\rho_0}\frac{\partial^2 P}{\partial x_i \partial x_j}(\mathbf{x},t) \tag{6.3}$$

and the incompressibility condition (6.2) yields

$$\mathrm{tr}\,(\Gamma) = \Gamma_{kk} = 0. \tag{6.4}$$

Now decompose $\Gamma = S + W$ into the sum of the rate of stretch tensor $S = \frac{1}{2}\left(\Gamma + \Gamma^T\right)$ and spin tensor $W = \frac{1}{2}\left(\Gamma - \Gamma^T\right)$ and substitute into (6.3) to yield

$$\frac{D}{Dt}\left(S_{ij} + W_{ij}\right) + \left(S^2 + W^2 + SW + WS\right)_{ij} = -\frac{1}{\rho_0}\frac{\partial^2 P}{\partial x_i \partial x_j}(\mathbf{x},t). \tag{6.5}$$

Next note that

$$\left(S^2 + W^2\right)^T = \left(S^T S^T + W^T W^T\right) = \left(S^2 + (-W)^2\right) = \left(S^2 + W^2\right)$$

and hence $\left(S^2 + W^2\right)$ is *symmetric.*

Also

$$(SW + WS)^T = \left(W^T S^T + S^T W^T\right) = ((-W)S + S(-W)) = -\left(SW + WS\right)$$

and hence $(SW + WS)$ is *skew symmetric.*

Now take the symmetric and skew-symmetric parts of (6.5) to obtain

$$\frac{DS_{ij}}{Dt} + \left(S^2 + W^2\right)_{ij} = -\frac{1}{\rho_0}\frac{\partial^2 P}{\partial x_i \partial x_j}(\mathbf{x},t) \tag{6.6}$$

and

$$\frac{DW_{ij}}{Dt} + (SW + WS)_{ij} = 0 \tag{6.7}$$

and the incompressibility condition (6.4) becomes

$$\mathrm{tr}S = S_{kk} = 0 \tag{6.8}$$

(since $\mathrm{tr}W = W_{kk} = 0$).

**Remark 6.1.** *Recall that the (skew-symmetric) spin tensor $W$ is related to the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}(\mathbf{x}, t)$ by*

$$\omega_i = (\nabla \times \boldsymbol{v})_i = \epsilon_{ijk}\left(\frac{\partial v_k}{\partial x_j}\right) = \frac{1}{2}\epsilon_{ijk}\frac{\partial v_k}{\partial x_j} - \frac{1}{2}\epsilon_{ikj}\frac{\partial v_k}{\partial x_j} = \epsilon_{ijk}\frac{1}{2}\left(\frac{\partial v_k}{\partial x_j} - \frac{\partial v_j}{\partial x_k}\right) = \epsilon_{ijk}W_{kj}.$$

*Conversely, since $2W\mathbf{a} = \boldsymbol{\omega} \times \mathbf{a}$, $\forall \mathbf{a} \in \mathbb{R}^3$, it follows that*

$$2W_{ij} = \epsilon_{ikj}\omega_k.$$

**Proposition 6.2.** *The system* (6.7) *is equivalent to the vorticity equation*

$$\frac{D\boldsymbol{\omega}}{Dt} = \Gamma\boldsymbol{\omega} = S\boldsymbol{\omega}. \tag{6.9}$$

*Proof.* The idea is to use Remark 6.1 to show that (6.7) implies (6.9) and to note that the steps in this argument can be reversed to show the converse implication.

Notice first that

$$\Gamma\boldsymbol{\omega} = (S + W)\boldsymbol{\omega} = S\boldsymbol{\omega} + W\boldsymbol{\omega} = S\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\omega} \times \boldsymbol{\omega} = S\boldsymbol{\omega}$$

and so the two expressions on the righthand side of (6.9) are clearly equivalent.

Now multiply expression (6.7) by $\epsilon_{mji}$ and use Remark 6.1 to obtain

$$\frac{D\omega_m}{Dt} + \epsilon_{mji}\,(SW + WS)_{ij} = 0$$

We next show that $\epsilon_{mji}\,(SW + WS)_{ij} = -(S\boldsymbol{\omega})_m$ which will complete the proof. To see this, use Remark 6.1 again to obtain

$$\begin{aligned}\epsilon_{mji}(SW)_{ij} &= \epsilon_{mji}\left(S_{ik}\left[\frac{1}{2}\epsilon_{knj}\omega_n\right]\right) = \frac{1}{2}\epsilon_{jim}\epsilon_{jkn}S_{ik}\omega_n = \frac{1}{2}\left[\delta_{ik}\delta_{mn} - \delta_{in}\delta_{mk}\right]S_{ik}\omega_n \\ &= \frac{1}{2}S_{ii}\omega_m - \frac{1}{2}S_{im}\omega_i = -\frac{1}{2}S_{mi}\omega_i = -\frac{1}{2}\,(S\boldsymbol{\omega})_m\ ,\end{aligned}$$

where we have used the symmetry of $S$ and (6.8).

A similar argument shows that

$$\epsilon_{mji}(WS)_{ij} = -\frac{1}{2}\,(S\boldsymbol{\omega})_m\,.$$

Hence we obtain that

$$\epsilon_{mji}\,(SW + WS)_{ij} = -\,(S\boldsymbol{\omega})_m\ ,$$

which completes the proof. □

**Theorem 6.3** (Constructing exact solutions.)**.** *Let $\bar{S}(t)$ be a real symmetric $3 \times 3$ matrix with $\mathrm{tr}(\bar{S}(t)) = 0$. Determine the vorticity $\bar{\boldsymbol{\omega}}$ from the system of ordinary differential equations*

$$\frac{d\bar{\boldsymbol{\omega}}(t)}{dt} = \bar{S}(t)\bar{\boldsymbol{\omega}}(t), \quad \bar{\boldsymbol{\omega}}(0) = \boldsymbol{\omega}_0 \in \mathbb{R}^3 \tag{6.10}$$

*(and the equivalent spin tensor from $2W_{ij} = \epsilon_{ikj}\bar{\omega}_k(t)$). Then*

$$\bar{\mathbf{v}}(\mathbf{x}, t) = \frac{1}{2}\bar{\boldsymbol{\omega}}(t) \times \mathbf{x} + \bar{S}(t)\mathbf{x} = \bar{W}(t)\mathbf{x} + \bar{S}(t)\mathbf{x} \tag{6.11}$$

*is an exact solution of the 3D Euler equations with corresponding pressure*

$$\bar{P}(\mathbf{x}, t) = -\frac{1}{2}x_i x_j\left[\dot{\bar{S}}_{ij} + \left(\bar{S}^2 + \bar{W}^2\right)_{ij}\right]\rho_0 \tag{6.12}$$

The proof of this result follows by:

1. Showing that $\bar{S}, \bar{W}, \bar{P}$ satisfy (6.6), (6.7),(6.8). In showing this, we use the fact that (6.7) is equivalent to the vorticity equations (6.9) by Proposition 6.2. We then note that (6.10) guarantees that (6.9) is satisfied since $\bar{\boldsymbol{\omega}}$ is independent of $\mathbf{x}$.

2. Having shown that (6.9) (and hence (6.7)) is satisfied, we then use (6.6) to define the corresponding pressure field. The assumption $\mathrm{tr}\bar{S} = 0$ guarantees that (6.8) is satisfied.

3. Since $\bar{S} + \bar{W} = \bar{\Gamma} = \nabla\bar{\mathbf{v}}$ is independent of $\mathbf{x}$ it will follow that

$$\bar{\mathbf{v}}(\mathbf{x}, t) = \bar{S}(t)\mathbf{x} + \bar{W}(t)\mathbf{x}$$

satisfies (6.1), (6.2).

**Remark 6.4.** *See problem sheet for an example of a flow of the form given in the last theorem. Notice also that in* (6.12) *the term* $x_i(S^2)_{ij}x_j = ||S\mathbf{x}||^2$ *and is therefore positive definite whilst, in contrast, the term* $x_i(W^2)_{ij}x_j = -||W\mathbf{x}||^2$ *is negative definite.*

# 7 Cartesian Tensors

We work in the three-dimensional Euclidean vector space $\mathbb{E}^3$. Let $\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a given (fixed) basis which is orthonormal with respect to the given inner-product $\langle \cdot, \cdot \rangle : \mathbb{E}^3 \times \mathbb{E}^3 \to \mathbb{R}$ on $\mathbb{E}^3$. (i.e. $\mathcal{A}$ is a basis for $\mathbb{E}^3$, and $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$ $\Rightarrow \|\mathbf{e}_i\| = \langle \mathbf{e}_i, \mathbf{e}_i \rangle^{\frac{1}{2}} = 1$ for $i = 1, 2, 3$).

A key principal in the modelling of continuous media (e.g. fluids, solids) is that the form and structure of properly formulated equations describing physical systems, and the relationship between physical quantities, should not depend on the choice of the cartesian coordinate system of the observer.

**Terminology**: A *cartesian frame* is a choice of origin together with a right-handed orthonormal basis/coordinate system.

The next few subsections recall the transformation properties of the components of a vector and of the matrix representing a linear transformation under a change of cartesian frame and will motivate our general treatment of cartesian tensors in section 7.4.

## 7.1 Matrix representation of a linear transformation

Let $\sigma : \mathbb{E}^3 \to \mathbb{E}^3$ be a linear transformation and let $\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, $\tilde{\mathcal{A}} = \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$ be bases for $\mathbb{E}^3$. Then the $(3 \times 3)$ matrix $A = (a_{ij}) \in M^{3\times 3}$ representing the linear transformation:

$$\sigma : (\mathbb{E}^3, \mathcal{A}) \to (\mathbb{E}^3, \tilde{\mathcal{A}})$$

is defined by the relation:

$$\sigma(\mathbf{e}_j) = a_{ij}\tilde{\mathbf{e}}_i$$

($j$ is a dummy index, and $i$ is a free index)

## 7.2 Effect of a Change of Orthonormal Basis on the Components Representing a Vector

Let $\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, $\tilde{\mathcal{A}} = \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$ be orthonormal bases for $\mathbb{E}^3$. Let $Q = (q_{ij}) \in M^{3\times 3}$ represent the identity map $id : (\mathbb{E}^3, \mathcal{A}) \to (\mathbb{E}^3, \tilde{\mathcal{A}})$ (i.e. $id(\mathbf{v}) = \mathbf{v}, \quad \forall \mathbf{v} \in \mathbb{E}^3$), i.e.,

$$id(\mathbf{e}_i) = \mathbf{e}_i = q_{ji}\tilde{\mathbf{e}}_j \qquad (*)$$

Then given $\mathbf{x} \in \mathbb{E}^3$, $\mathbf{x} = x_i\mathbf{e}_i = x_i(q_{ji}\tilde{\mathbf{e}}_j) = (q_{ji}x_i)\tilde{\mathbf{e}}_j = \tilde{x}_j\tilde{\mathbf{e}}_j$, where

$$\tilde{x}_j = q_{ji}x_i \iff \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = Q \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

**Proposition 7.1.** *The change of bases matrix $Q$ above is an orthogonal matrix.*

*Proof.*

$$\left.\begin{array}{ll}\text{if } i \neq j & 0 \\ \text{if } i = j & 1\end{array}\right\} = \langle \mathbf{e}_i, \mathbf{e}_j \rangle = \langle q_{li}\tilde{\mathbf{e}}_l, q_{kj}\tilde{\mathbf{e}}_k \rangle \quad \text{by (*)}$$
$$\begin{aligned} &= q_{li}\langle \tilde{\mathbf{e}}_l, q_{kj}\tilde{\mathbf{e}}_k \rangle \\ &= q_{li}q_{kj}\langle \tilde{\mathbf{e}}_l, \tilde{\mathbf{e}}_k \rangle \\ &= q_{ki}q_{kj} \quad \text{(only non-zero terms when } l = k) \\ &= (Q^T)_{ik}(Q)_{kj} = (Q^TQ)_{ij} \end{aligned}$$

□

**Remark 7.2.** *Notice from the proof (see (*)) that for each $i$, $\langle \mathbf{e}_i, \tilde{\mathbf{e}}_k \rangle = \langle q_{ji}\tilde{\mathbf{e}}_j, \tilde{\mathbf{e}}_k \rangle = q_{ji}\langle \tilde{\mathbf{e}}_j, \tilde{\mathbf{e}}_k \rangle = q_{ki}$. Hence the $i^{th}-$column of $Q$ consists of the components of the $\mathbf{e}_i$ in the basis $\tilde{\mathcal{A}}$. Similarly, the $j^{th}$ row of $Q$ consists of the components of $\tilde{\mathbf{e}}_j$ in the bases $\mathcal{A}$. Hence, $\tilde{\mathbf{e}}_j = q_{jk}\mathbf{e}_k$.*

### Terminology

If the matrix $Q \in O(3)$ (orthogonal $3 \times 3$ matrices) satisfies $\det Q = 1$ (i.e., if $Q \in SO(3)$- the group of special orthogonal matrices), then $\tilde{\mathcal{A}}$ is said to be oriented in the same sense as $\mathcal{A}$.
If $\det Q = -1$, then we say that $\tilde{\mathcal{A}}$ is oriented in the opposite sense to $\mathcal{A}$.
If $\tilde{\mathcal{A}}$ , $\mathcal{A}$ are oriented in the same sense, then there is a rotation mapping $\mathbf{e}_i \to \tilde{\mathbf{e}}_i$.
If they are oriented in opposite senses, then we need a rotation and a reflection in general.

## 7.3 Effect of Change of Orthonormal Basis on the Matrix Representing a Linear Transformation

Let $\sigma : \mathbb{E}^3 \to \mathbb{E}^3$ be a linear transformation. Let $\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, $\tilde{\mathcal{A}} = \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$ be orthonormal bases for $\mathbb{E}^3$. Let $A \in M^{3\times 3}$ represents $\sigma : (\mathbb{E}^3, \mathcal{A}) \to (\mathbb{E}^3, \mathcal{A})$. Let $\tilde{A} \in M^{3\times 3}$ represent $\sigma : (\mathbb{E}^3, \tilde{\mathcal{A}}) \to (\mathbb{E}^3, \tilde{\mathcal{A}})$. Let $Q \in O(3)$ represents $id : (\mathbb{E}^3, \mathcal{A}) \to (\mathbb{E}^3, \tilde{\mathcal{A}})$.

$$\begin{tikzcd} (\mathbb{E}^3, \mathcal{A}) \arrow[r, "\sigma;A"] \arrow[d, "id;Q"'] & (\mathbb{E}^3, \mathcal{A}) \\ (\mathbb{E}^3, \tilde{\mathcal{A}}) \arrow[r, "\sigma;\tilde{A}"'] & (\mathbb{E}^3, \tilde{\mathcal{A}}) \arrow[u, "id;Q^T"'] \end{tikzcd}$$

Hence $\sigma = id \text{ o } \sigma \text{ o } id$ is represented by $A = Q^T\tilde{A}Q \iff \tilde{A} = QAQ^T$. Hence,

$$\tilde{a}_{ij} = q_{ik}a_{kl}q_{jl} = q_{ik}q_{jl}a_{kl}$$

## 7.4 Cartesian Tensors

From now on we work only with Cartesian frames (i.e., with coordinates corresponding to right-handed orthonormal bases for $\mathbb{E}^3$).

In this case, suppose that $x_1, x_2, x_3$ and $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ are coordinates corresponding to the (right-handed) orthonormal bases $\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\tilde{\mathcal{A}} = \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$, where $\tilde{\mathbf{e}}_i = q_{ij}\mathbf{e}_j$,

$Q = (q_{ij}) \in SO(3)$. Then we have seen that the $3^1 = 3$ components $x_1, x_2, x_3$ of a vector $\mathbf{x} \in \mathbb{E}^3$ in the basis $\mathcal{A}$ transform according to

$$\tilde{x}_i = q_{ij} x_j,$$

under the change of basis from $\mathcal{A}$ to $\tilde{\mathcal{A}}$.

Similarly, the $3^2 = 9$ entries in the matrix $A = (a_{ij}) \in M^{3\times 3}$ representing the linear transformation $\sigma : (\mathbb{E}^3, \mathcal{A}) \to (\mathbb{E}^3, \mathcal{A})$ transform according to

$$\tilde{a}_{ij} = q_{ik} q_{jl}\, a_{kl},$$

where $\tilde{A} = (\tilde{a}_{ij}) \in M^{3\times 3}$ represents $\sigma : (\mathbb{E}^3, \tilde{\mathcal{A}}) \to (\mathbb{E}^3, \tilde{\mathcal{A}})$.

**Definition (Cartesian Tensor)**

A *cartesian tensor* $\underline{\underline{T}}$ *of order* $n$ is represented in a cartesian frame, with right-handed orthonormal basis $\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for $\mathbb{E}^3$, by $3^n$ associated components, denoted $T_{i_1 i_2 \ldots i_n}$ ($n$-indices), where each suffix takes the value $1, 2$ or $3$.

These elements transform according to

$$\tilde{T}_{i_1 i_2 \ldots i_n} = q_{i_1 j_1} q_{i_2 j_2} \ldots q_{i_n j_n} T_{j_1 j_2 \ldots j_n}$$

for each $Q = (q_{ij}) \in SO(3)$, where $\tilde{T}_{i_1 i_2 \ldots i_n}$ are the components of the tensor $\underline{\underline{T}}$ in the right-handed orthonormal basis $\tilde{\mathcal{A}} = \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$, with $\tilde{\mathbf{e}}_i = q_{ij} \mathbf{e}_j$.

**Remarks**

1. The term cartesian tensor of order n is sometimes abbreviated to CTn.

2. We can think of the components of $\underline{\underline{T}}$ as the entries in a multi-dimensional matrix.

3. The (confusing) terminology "cartesian tensor of rank n" is also sometimes used in the literature.

**Examples.**

**1.** A CT0 is called a *scalar invariant.*

(i) Any constant $\alpha \in \mathbb{R}$ is clearly a CT0 (since $\tilde{\alpha} = \alpha$).

(ii) For $\mathbf{x} = x_i \mathbf{e}_i, \mathbf{y} = y_j \mathbf{e}_j \in \mathbb{E}^3$, the expression $x_i y_i$ $(= \langle \mathbf{x}, \mathbf{y} \rangle)$ is a scalar invariant.

**Proof.** For $Q = (q_{ij}) \in SO(3)$,

$$\tilde{x}_i \tilde{y}_i = (q_{ik} x_k)(q_{il} y_l) = q_{ik} q_{il} x_k y_l = (Q^T Q)_{kl} x_k x_l = x_k y_k.$$

(iii) Let $\phi : [0, \infty) \to \mathbb{R}$, then $\phi\left((x_1)^2 + (x_2)^2 + (x_3)^2\right)$ is a scalar invariant (this follows from (ii) above on setting $\mathbf{y} = \mathbf{x}$).

**2.** A CT1

(i) Any vector $\mathbf{x} = x_i \mathbf{e}_i \in \mathbb{E}^3$ is a CT1 since its components transform according to $\tilde{x}_i = q_{ij} x_j$, $Q = (q_{ij}) \in SO(3)$ under the change of cartesian frame from $\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\tilde{\mathcal{A}} = \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$, $\tilde{\mathbf{e}}_i = q_{ij} \mathbf{e}_j$.

(ii) Consider a smooth function $\phi(x_1, x_2, x_3)$ of the coordinates in the basis $\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. We define the gradient of $\phi$ by $\nabla\phi = \frac{\partial \phi}{\partial x_i} \mathbf{e}_i$, then $\nabla \phi$ is a CT1. (See problem sheet 1.)

**3.** A CT2

Any linear transformation $\sigma : \mathbb{E}^3 \to \mathbb{E}^3$ corresponds to a CT2, since the $3^2$ entries in the matrix $A = (a_{ij}) \in M^{3\times 3}$ representing $\sigma : (\mathbb{E}^3, \mathcal{A}) \to (\mathbb{E}^3, \mathcal{A})$ in the right-handed orthonormal basis $\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ transform according to $\tilde{a}_{ij} = q_{ik} q_{jl} a_{kl}$ for $Q = (q_{ij}) \in SO(3)$ under the change of cartesian basis from $\mathcal{A}$ to $\tilde{\mathcal{A}} = \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$, where $\tilde{\mathbf{e}}_i = q_{ik} \mathbf{e}_k$.

## 7.5 Examples of cartesian tensors

### Kronecker Tensor

Recall that the Kronecker delta denoted $\delta_{ij}$ is defined by

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} .$$

The **Kronecker tensor** is a CT2 defined, in any cartesian frame $\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ by the components $T_{ij} = \langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}$. To see that it is a CT2, notice that if $\tilde{\mathcal{A}} = \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$ corresponds to another cartesian frame with $\tilde{\mathbf{e}}_i = q_{ij}\mathbf{e}_j$, $Q = (q_{ij}) \in SO(3)$, then

$$\tilde{T}_{ij} = \langle \tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j \rangle = \langle q_{ik}\mathbf{e}_k, q_{jl}\mathbf{e}_l \rangle = q_{ik}q_{jl}\langle \mathbf{e}_k, \mathbf{e}_l \rangle = q_{ik}q_{jl}T_{kl}.$$

Moreover, since

$$q_{ik}q_{jl}T_{kl} = q_{ik}q_{jl}\delta_{kl} = q_{ik}q_{jk} = \delta_{ij} = T_{ij},$$

it further follows that $\tilde{T}_{ij} = T_{ij}$ and we say that this tensor is isotropic.

**Definition.** A tensor is *isotropic* if all its components are unchanged under a change of cartesian frame.

In fact, it can be shown that, up to multiplication by a scalar, this is the only isotropic CT2.

### The Alternating Tensor

Recall the alternating symbol $\epsilon_{ijk}$ defined earlier in the course. The **alternating tensor** is a CT3, defined in the cartesian frame $\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ by the components

$$T_{ijk} = \langle \mathbf{e}_i, \mathbf{e}_j \times \mathbf{e}_k \rangle = \epsilon_{ijk}.$$

To see that it is a CT3, notice that if $\tilde{\mathcal{A}} = \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$ corresponds to another cartesian frame with $\tilde{\mathbf{e}}_i = q_{ij}\mathbf{e}_j$, $Q = (q_{ij}) \in SO(3)$, then

$$\tilde{T}_{ijk} = \langle \tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j \times \tilde{\mathbf{e}}_k \rangle = \langle (q_{il}\mathbf{e}_l), (q_{jm}\mathbf{e}_m \times q_{kn}\mathbf{e}_n) \rangle = q_{il}q_{jm}q_{kn}\langle \mathbf{e}_l, \mathbf{e}_m \times \mathbf{e}_n \rangle = q_{il}\, q_{jm}q_{kn}T_{lmn},$$

so the $T_{ijk}$ are elements of a CT3.

Moreover, since

$$q_{il}q_{jm}q_{kn}T_{lmn} = q_{il}q_{jm}q_{kn}\epsilon_{lmn} = \epsilon_{ijk} = T_{ijk},$$

it further follows that $\tilde{T}_{ijk} = T_{ijk}$ and so this tensor is *isotropic*. It can be shown that, up to multiplication by a scalar, this is the only isotropic CT3.

## 7.6 Elementary Operations with Cartesian Tensors

### Linear Combinations.

Let $\underline{\underline{S}}$ , $\underline{\underline{T}}$ be CTn's. Then, given $\alpha, \beta \in \mathbb{R}$, we define $\left(\alpha\underline{\underline{S}} + \beta\underline{\underline{T}}\right)_{i_1..i_n} = \alpha S_{i_1...i_n} + \beta T_{i_1...i_n}$. It is easily verified that the left hand side terms are components of a CTn.

**Contraction of Indices.**

Let $\underline{\underline{T}}$ be a CTn with components $T_{ijk\ldots}$. If we set any two of the free indices equal (thereby effecting a sum) then the resulting elements are components of a CT(n-2).

**Proof.** Since $\underline{\underline{T}}$ is a CTn, given $(q_{ij}) \in SO(3)$, it follows that

$$\tilde{T}_{ijkl\ldots} = q_{ir}q_{js}q_{kt}q_{lu}\ldots..T_{rstu\ldots} \ .$$

Now set $j = k$ to obtain

$$\tilde{T}_{ikkl\ldots} = q_{ir}q_{ks}q_{kt}q_{lu}\ldots T_{rstu\ldots} = q_{ir}q_{lu}\ldots T_{rkku\ldots} \quad (\text{since } q_{ks}q_{kt} = (Q^T Q)_{st} = \delta_{st}\ )$$

Hence, $S_{il\ldots} = T_{ikkl\ldots}$ are the components of a CT(n-2).

**Example.**

Let $T_{ij}$ be components of $\underline{\underline{T}}$ which is a CT2. Then $T_{ii}$ is called the *trace* of $\underline{\underline{T}}$, and the trace of $\underline{\underline{T}}$ is a CT0 (i.e., a scalar invariant).

**Product of Cartesian Tensors.**

If $\underline{\underline{T}}$ is a CTn and $\underline{\underline{S}}$ is a CTm, then defining

$$U_{i_1\ldots i_{(n+m)}} = T_{i_1\ldots i_n} S_{i_{(n+1)}\ldots i_{(n+m)}}$$

yields the components of a CT(n+m).

**Proof.** Let $Q = (q_{ij}) \in SO(3)$, then

$$q_{i_1 j_1} \cdots q_{i_{(n+m)} j_{(n+m)}} U_{j_1\ldots j_{(n+m)}} = q_{i_1 j_1} \cdots q_{i_n j_n} T_{j_1\ldots j_n}\ \ q_{i_{(n+1)} j_{(n+1)}} \cdots . \ q_{i_{(n+m)} j_{(n+m)}} S_{j_{(n+1)}\ldots j_{(n+m)}}$$

and, since $\underline{\underline{S}}$ , $\underline{\underline{T}}$ are both cartesian tensors it follows that this equals

$$\tilde{T}_{i_1\ldots i_n} \tilde{S}_{i_{(n+1)}\ldots i_{(n+m)}} = \tilde{U}_{i_1\ldots i_{(n+m)}}.$$

Hence $U_{i_1\ldots i_{(n+m)}}$ are components of a CT(n+m).

**Tensor Gradient.** Suppose that $T_{i_1\ldots i_n}$ are components of $\underline{\underline{T}}$, a CTn. Suppose further that these components are functions of the coordinates $x_1, x_2, x_3$ in the cartesian basis $\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Define

$$S_{i_1\ldots i_n k} = \frac{\partial}{\partial x_k} T_{i_1\ldots i_n}.$$

Then $S_{i_1\ldots i_n k}$ are components of a CT(n+1).

**Proof.** Since $\underline{\underline{T}}$ is a CTn, $\tilde{T}_{i_1\ldots i_n} = q_{i_1 j_1}\ldots q_{i_n j_n} T_{j_1 j_2\ldots j_n}$ for any $Q = (q_{ij}) \in SO(3)$. Hence, by the chain rule,

$$\frac{\partial}{\partial \tilde{x}_k} \tilde{T}_{i_1\ldots i_n} = q_{i_1 j_1}\ldots q_{i_n j_n} \frac{\partial}{\partial x_m} \left(T_{j_1 j_2\ldots j_n}\right) \frac{\partial x_m}{\partial \tilde{x}_k}.$$

However, $\tilde{x}_k = q_{kl}x_l \Leftrightarrow x_m = q_{km}\tilde{x}_k$, so $\frac{\partial x_m}{\partial \tilde{x}_k} = q_{km}$ and hence

$$\frac{\partial}{\partial \tilde{x}_k}\tilde{T}_{i_1 \dots i_n} = q_{i_1 j_1} \dots q_{i_n j_n} q_{km} \frac{\partial}{\partial x_m} \left(T_{j_1 j_2 \dots j_n}\right).$$

Hence $\tilde{S}_{i_1 \dots i_n k} = q_{i_1 j_1} \dots q_{i_n j_n} q_{km} S_{j_1 \dots j_n m}$ which shows that $S_{i_1 \dots i_n m}$ are components of a CT(n+1).

**Remark.**(Differentiating with respect to a time-like parameter.) Suppose that the components $T_{i_1 \dots i_n}$ of a CTn depend not only on the components $x_1, x_2, x_3$ in the cartesian frame basis $\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ but also on an independent parameter $t$ (e.g., representing time). Then for each $(q_{ij}) \in SO(3)$ we have that

$$\frac{d}{dt}\left(\tilde{T}_{i_1 \dots i_n}\right) = q_{i_1 j_1} \dots q_{i_n j_n} \frac{d}{dt}\left(T_{i_1 \dots i_n}\right)$$

and so

$$\dot{T}_{i_1 \dots i_n} = \frac{d}{dt}\left(T_{i_1 \dots i_n}\right)$$

are the components of a CTn.

## Symmetric and Skew (Anti) Symmetric Cartesian Tensors of order 2

If $\underline{\underline{T}}$ is a CT2 with components $T_{ij}$ in the cartesian basis $\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, then $\underline{\underline{T}}$ is said to be symmetric if $T_{ij} = T_{ji}$.

Notice in this case, $\tilde{T}_{ij} = q_{il}q_{jm}T_{lm} = q_{il}q_{jm}T_{ml} = q_{jm}q_{il}T_{ml} = \tilde{T}_{ji}$, for $Q = (q_{ij}) \in SO(3)$ and so symmetry is independent of the choice of cartesian coordinate frame.

Similarly, $\underline{\underline{T}}$ is said to be skew (or anti) symmetric if $T_{ij} = -T_{ji}$. Again this definition can be shown to be independent of the choice of cartesian coordinate frame.

The components of any CT2 $\underline{\underline{T}}$, can be expressed as $T_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) + \frac{1}{2}(T_{ij} - T_{ji}) = E_{ij} + W_{ij}$, where $E_{ij}, W_{ij}$ are the components of a symmetric tensor $\mathbf{E}$ and skew symmetric tensor $\mathbf{W}$.

**The Quotient Rule for Cartesian Tensors.**

**Special Case.**

Suppose that a mathematical object has the property that, given any cartesian frame (using components $x_1, x_2, x_3$, in the right-handed orthonormal basis $\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$) it can be represented by $3^2 = 9$ quantities $a_{ij}$ (which will in general be given by $\tilde{a}_{ij}$ in a different cartesian frame using components $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ in the right-handed orthonormal basis $\tilde{\mathcal{A}} = \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$). Suppose further that for all vectors (i.e, any CT1) $\mathbf{v} = v_i \mathbf{e}_i \in \mathbb{E}^3$ the $w_i = a_{ij} v_j$ are the components of a vector (i.e., a CT1). Then the $a_{ij}$ are the components of a CT2.

**Proof.**

Suppose that $Q = (q_{ij}) \in SO(3)$ and that under the corresponding change of cartesian frame $\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \to \tilde{\mathcal{A}} = \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$ we have $a_{ij} \to \tilde{a}_{ij},\ v_i \to \tilde{v}_i,\ w_i \to \tilde{w}_i$. Then, by assumption,

$$\tilde{a}_{ij} \tilde{v}_j = \tilde{w}_i \quad .$$

Hence, since $\underline{v}$, $\underline{\underline{w}}$ are cartesian tensors, it follows that

$$\tilde{a}_{ij} \left( q_{jl} v_l \right) = q_{im} w_m = q_{im} \left( a_{mk} v_k \right) .$$

Thus

$$\left( \tilde{a}_{ij} q_{jl} - q_{im} a_{ml} \right) v_l = 0$$

for any vector $\mathbf{v}$, and so

$$\tilde{a}_{ij} q_{jl} = q_{im} a_{ml} .$$

Hence, multiplying both sides by $q_{tl}$ yields

$$\tilde{a}_{ij} q_{jl} q_{tl} = q_{im} q_{tl} a_{ml} \Rightarrow \tilde{a}_{it} = q_{im} q_{tl} a_{ml} \quad \text{(since } \ q_{jl} q_{tl} = \delta_{jt}).$$

Thus the elements $a_{ij}$ are the elements of a CT2.

## The Quotient Rule for Cartesian Tensors (General Version)

Suppose that a mathematical object has the property that, for any given Cartesian coordinate system (coordinates $X_1, X_2, X_3$, in the right-handed orthonormal basis $\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$), it can be represented by $3^{n+m}$ quantities

$$a_{i_1 i_2 \ldots i_m j_1 j_2 \ldots j_n} \qquad (\dagger)$$

(which will in general vary according to the chosen basis). Suppose further that for any CTn $\mathbf{V}$, with components $V_{j_1 \ldots j_n}$, the quantities

$$W_{i_1 \ldots i_m} = a_{i_1 \ldots i_m j_1 \ldots j_n} V_{j_1 \ldots j_n}$$

are the components of a CTm. Then the $a_{i_1 \ldots i_m j_1 \ldots j_n}$ are the components of a CT(n+m).

*Proof.* Consider a change of Cartesian frame corresponding to $(q_{ij}) \in SO(3)$, from the Cartesian basis $\mathcal{A}$ to the Cartesian basis $\tilde{\mathcal{A}} = \{\tilde{\mathbf{e_1}}, \tilde{\mathbf{e_2}}, \tilde{\mathbf{e_3}}\}$. Then the expression ($\dagger$) transforms to

$$\tilde{a}_{i_1 \ldots i_m j_1 \ldots j_n} \tilde{V}_{j_1 \ldots j_n} = \tilde{W}_{i_1 \ldots i_m} = q_{i_1 j_1} q_{i_2 j_2} \ldots q_{i_m j_m} W_{j_1 \ldots j_m}$$

since $\mathbf{W}$ is a CTm by assumption. Hence, since $\mathbf{V}$ is also a CTn, it follows using ($\dagger$) that

$$\tilde{a}_{i_1 \ldots i_m j_1 \ldots j_n} q_{j_1 k_1} q_{j_2 k_2} \ldots q_{j_n k_n} V_{k_1 \ldots k_n} = q_{i_1 j_1} q_{i_2 j_2} \ldots q_{i_m j_m} W_{j_1 \ldots j_m}$$
$$\Rightarrow \tilde{a}_{i_1 \ldots i_m j_1 \ldots j_n} q_{j_1 k_1} q_{j_2 k_2} \ldots q_{j_n k_n} V_{k_1 \ldots k_n} = q_{i_1 j_1} q_{i_2 j_2} \ldots q_{i_m j_m} a_{j_1 \ldots j_m k_1 \ldots k_n} V_{k_1 \ldots k_n}$$

Hence

$$[\tilde{a}_{i_1 \ldots i_m j_1 \ldots j_n} q_{j_1 k_1} q_{j_2 k_2} \ldots q_{j_n k_n} - q_{i_1 j_1} q_{i_2 j_2} \ldots q_{i_m j_m} a_{j_1 \ldots j_m k_1 \ldots k_n}] V_{k_1 \ldots k_n} = 0$$

holds for all choices of $V_{k_1 \ldots k_n}$ and hence

$$[\tilde{a}_{i_1 \ldots i_m j_1 \ldots j_n} q_{j_1 k_1} q_{j_2 k_2} \ldots q_{j_n k_n} - q_{i_1 j_1} q_{i_2 j_2} \ldots q_{i_m j_m} a_{j_1 \ldots j_m k_1 \ldots k_n}] = 0$$

Multiplying the above expression by $q_{t_1 k_1} \ldots q_{t_n k_n}$ (and using the summation convention and noting that $q_{t_1 k_1} q_{j_1 k_1} = \delta_{t_1 j_1}, q_{t_2 k_2} q_{j_2 k_2} = \delta_{t_2 j_2}$etc. ) we obtain

$$\tilde{a}_{i_1 \ldots i_m t_1 \ldots t_n} q_{i_1 j_1} q_{i_2 j_2} \ldots q_{i_m j_m} q_{t_1 k_1} \ldots q_{t_n k_n} a_{j_1 \ldots j_m k_1 \ldots k_n} = 0$$

or, equivalently, $\tilde{a}_{i_1 \ldots i_m t_1 \ldots t_n} = q_{i_1 j_1} q_{i_2 j_2} \ldots q_{i_m j_m} q_{t_1 k_1} \ldots q_{t_n k_n} a_{j_1 \ldots j_m k_1 \ldots k_n}$, which proves that that the $a_{i_1 \ldots i_m t_1 \ldots t_n}$ are the components of a CT(m+n). $\square$

### Example:

$n = 1, m = 0$, $\mathbf{x} \times \mathbf{y} = \epsilon_{ijk} x_j y_k \mathbf{e}_i$. Given any $\mathbf{z} = z_i \mathbf{e}_i \in \mathbb{E}^3$, then let $a_i = \epsilon_{ijk} x_j y_k$. Then $a_i z_i = \begin{vmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$ is a CT0 (see sheet 2, Q1). Hence, by the quotient rule, the $a_i$ are components of a CT1, i.e. $\mathbf{x} \times \mathbf{y}$ is a vector.

# 8 Basic results from Linear Algebra

**Theorem 8.1.** *Let $A \in M^{3\times 3}$ be symmetric (i.e. $A^T = A$). Then, we have:*

1) *All eigenvalues of $A$ are real, we define these as $\lambda_1, \lambda_2, \lambda_3$ (possibly repeated)*

2) *There is an orthonormal basis for $\mathbb{R}^3$ consisting of corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ (i.e. $A\mathbf{v}_i = \lambda_i \mathbf{v}_i, \quad \langle \mathbf{v}_i, \mathbf{v}_j \rangle = \mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij}$).*

**Proposition 8.2.** *If $A \in M^{3\times 3}$ is symmetric with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and corresponding orthonormal eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Then $\exists Q \in SO(3)$ such that $Q^T A Q = diag(\lambda_1, \lambda_2, \lambda_3) =: D$.*

Sketch proof: Define $Q := [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$. Then $AQ = [A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3] = [\lambda_1\mathbf{v}_1, \lambda_2\mathbf{v}_2, \lambda_3\mathbf{v}_3] = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = D$.

Now $Q^T Q = I$ and so $Q^T A Q = D$. Without loss of generality, $Q \in SO(3)$ (if $\det Q = -1$, then replace $\mathbf{v}_1$ with $-\mathbf{v}_1$)

**Theorem 8.3** (Spectral Decomposition Theorem)**.** *Let $A \in M^{3\times 3}$ be symmetric with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be an orthonormal basis for $\mathbb{R}^3$ of corresponding eigenvectors. Then $A = \lambda_1 \mathbf{v}_1 \otimes \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 \otimes \mathbf{v}_2 + \lambda_3 \mathbf{v}_3 \otimes \mathbf{v}_3$. (Note: $(\mathbf{c} \otimes \mathbf{d})(\mathbf{x}) = \langle \mathbf{d}, \mathbf{x} \rangle \mathbf{c}$.)*

*Proof.* If $\mathbf{x} \in \mathbb{R}^3$, then $\mathbf{x} = \sum_{i=1}^3 \langle \mathbf{x}, \mathbf{v}_i \rangle \mathbf{v}_i$. Hence

$$A\mathbf{x} = \sum_{i=1}^3 \langle \mathbf{x}, \mathbf{v}_i \rangle A\mathbf{v}_i = \sum_{i=1}^3 \lambda_i \langle \mathbf{x}, \mathbf{v}_i \rangle \mathbf{v}_i = \sum_{i=1}^3 \lambda_i (\mathbf{v}_i \otimes \mathbf{v}_i)(\mathbf{x}) \tag{8.1}$$

$$= (\sum_{i=1}^3 \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i)(\mathbf{x}) \tag{8.2}$$

Since $\mathbf{x}$ was arbitrary, the result follows. □

**Corollary 8.4.** *For $k \in \mathbb{N}, A^k = \sum_{i=1}^3 \lambda_i^k \mathbf{v}_i \otimes \mathbf{v}_i$. This representation extends to $k \in \mathbb{Z}$ provided $\lambda_i \neq 0, \quad \forall i = 1, 2, 3$.*

**Theorem 8.5** (Square root theorem)**.** *Let $C$ be a symmetric, positive definite $n \times n$ matrix. Then there exists a unique positive definite symmetric matrix $U$ such that $C = U^2$. (We write $C^{\frac{1}{2}} = U$.)*

*Proof.* Let

$$C = \sum_{i=1}^n \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i$$

be the spectral decomposition of $C$. Notice that since $C$ is positive-definite, all its eigenvalues are positive. Now define

$$U = \sum_{i=1}^n \lambda_i^{\frac{1}{2}} \mathbf{v}_i \otimes \mathbf{v}_i,$$

then $U$ is symmetric and positive-definite and satisfies $U^2 = C$. □

**Theorem 8.6** (Polar decomposition theorem)**.** *Let $F$ satisfy* $\det F > 0$*. Then there exists $R \in SO(n)$ and positive definite symmetric matrices $U, V$ such that $F = RU = VR$.*

*Proof.* Let $C = F^T F$, then $C$ is symmetric since $C^T = (F^T F)^T = F^T F = C$ and so by the square root theorem, there exists a unique symmetric positive-definite square root $U$ (so that $C = U^2$).

Now define $R = FU^{-1}$, then

$$R^T R = (FU^{-1})^T (FU^{-1}) = U^{-T} F^T F U^{-1} = U^{-1} C U^{-1} = I$$

So $R$ is orthogonal and $R \in SO(n)$. By construction, we see that this decomposition is unique.

A similar proof working with $C = F F^T$ yields the (unique) decomposition $F = V\tilde{R}$ with $V$ positive-definite and symmetric and $\tilde{R} \in SO(n)$. To see that $\tilde{R} = R$, notice that $F = \tilde{R}\tilde{R}^T V \tilde{R}$, where $\tilde{R}^T V \tilde{R}$ is positive-definite and symmetric and so by the uniqueness of the decomposition $C = RU$ it follows that $R = \tilde{R}$ and $U = \tilde{R}^T V \tilde{R}$. $\square$